Effect of Periodic Roll Rate in Missiles with Proportional Navigation

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In several homing missiles, detrimental effects of excessive roll rate on the homing performance have been experienced. Flight tests have indicated that frequently the roll rate exhibits an almost periodic behavior. In this paper, the homing of a missile, guided by proportional navigation, with periodic roll rate, is solved in a closed form. As the first phase of the solution, the steady-state step response of a two-channel system with periodic roll rate is calculated. Approximate, but fairly accurate, simple formulae obtained by Fourier expansion provide insight into the effects induced by the periodic roll rate component. From the closed form solution, the necessary condition for convergent line of sight rate is derived. This condition indicates that the limitation on the average value of the periodic roll rate is more severe than for a constant roll rate.

Nomenclature‡

a	= effective proportional navigation constant
$a_n(t)$	= time-varying coefficient defined in Eq. (17)
A_i	= $n\times n$ constant matrix in Eq. (1)
$\mathbf{A}$	= $2n\times 2n$ constant matrix in Eq. (2)
$\underline{b}_i$	= $n\times 1$ constant column vector in Eq. (1)
$b_n(t)$	= time varying coefficient defined in Eq. (17)
$\mathbf{B}$	= $2n\times 2$ constant matrix in Eq. (2)
c_i^T	= $1\times n$ constant row vector in Eq. (1)
C	= constant cross-coupling factor defined by Eq. (20)
$C(t)$	= time-varying cross-coupling function defined in Eq. (25)
$\mathbf{C}$	= $2\times 2n$ constant matrix in Eq. (2)
$C_0,C_1,C_2...C_n$	= coefficients in the Fourier expansion of $C(t)$ in Eq. (31)
$D_1,D_2...D_n$	= coefficients in the Fourier expansion of $C(t)$ in Eq. (31)
$\mathfrak{D}(t)$	= 2×2 time varying matrix in Eq. (42)
$E_1(t), E_2(t)$	= time-varying functions defined in Eq. (43)
$\underline{g}_n(t)$	= time varying vector function defined in Eq. (15)
G	= constant direct gain defined in Eq. (15)
$G(t)$	= time varying direct gain function defined in Eq. (25)
$G_0,G_1,G_2...G_n$	= coefficient in the Fourier expansion of $G(t)$ in Eq. (30)
$h(t)$	= impulse response of each guidance channel in body coordinates
h_n	= constant defined in Eq. (22)
$\mathbf{I}$	= two-dimensional identity matrix
$\mathbf{J}$	= two-dimensional matrix defined by Eq. (6)
k_0	= constant defined in Eq. (A48)
$K_1,K_2,...K_n$	= coefficient in the Fourier expansion of $G(t)$ in Eq. (30)
n	= running index
s	= Laplace variable
t	= time
t_0	= missile time of flight till intercept
$\underline{v}$	= two-dimensional input vector
$\underline{w}$	= two-dimensional output vector
$\underline{x}_i$	= $n\times 1$ state vector in Eq. (1)
$\underline{x}$	= $2n\times 1$ state vector in Eq. (2)
y,z	= fixed directions in the plane normal to missile longitudinal axis
x_0,x_n	= variables defined in Eq. (A41)
$\dot{\alpha}_\phi$	= constant roll rate
α^*	= nondimensional constant roll rate defined in Eq. (33)
$\dot{\beta}_\phi$	= amplitude of periodic roll rate
β^*	= nondimensional periodic roll rate defined in Eq. (33)
δ_n	= constant defined in Eq. (A41).
ϵ_n,ϵ^*	= positive small quantities
λ	= angle between the line of sight and the line of reference (see Fig. 7)
$\dot{\underline{\lambda}}$	= two-dimensional line of sight rate vector
ξ	= dummy variable
ϕ	= missile roll angle
ϕ_p	= amplitude of roll oscillations defined in Eq. (38)
$\Phi(t)$	= 2×2 rotation matrix
$\Phi_n(t)$	= 2×2 matrix defined by Eq. (16)
ψ_ϕ	= phase angle of the periodic roll rate in Eq. (3)
ψ_n	= phase angle in the Fourier expansion of $G(t)$ in Eq. (30)
χ_n	= phase angle in the Fourier expansion of $C(t)$ in Eq. (31)
τ_1	= first order time constant
ω_ϕ	= frequency of the periodic roll rate
ω^*	= non-dimensional frequency defined in Eq. (33)

Received June 30, 1975; revision received Feb. 5, 1976.

Index categories: Navigation, Control, and Guidance Theory; LV/M Dynamics and Control; LV/M Guidance Systems (including Command and Information Systems).

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‡Vectors in body axes are denoted by a bar (̄), while vectors in nonrolling coordinates are underlined (_). Matrices are bold face capital letters.

Introduction

THE interaction of rolling motion with missile performance has two entirely different aspects. For finned missiles of all types (guided and unguided), high roll rate can seriously interfere with dynamic stability. Roll resonance and Magnus effects have been the topics of investigation since the early fifties,[1-4] and research activity in this field is still going on.[5,6]

For guided missiles, however, the roll effects are far more complex. Even for relatively "slow" roll rates, which do not create any problem of dynamic stability, missile performance deteriorates. It is specially true for cruciform homing missiles, which being insensitive to roll position (with the exception of those which have linearly polarized antenna patterns), generally do not have positional roll control. It has been experienced, however, that for effective homing, high roll rates must be avoided, either by a roll rate control system or by aeromechanical damping devices called "rollerons."[7,8] The problem of the roll rate effects on homing guidance has been investigated for some years, and for constant roll rates, an approximate closed form solution of the problem was obtained.[9-12] This solution provides an insight into the phenomenon of roll-induced cross-coupling in homing missiles and has established a simple and useful criterion to determine the highest permissible constant roll rate. Simulations of three-dimensional real missile models confirmed that if the roll rate exceeds the value predicted by the approximate theory, the missile trajectory will diverge.

Analysis of flight test data[13] indicated, however, that the roll rate of some homing missiles may very often have an important "almost periodic" component (in addition to a constant roll rate considered in the first closed form solution). The frequency of the roll rate oscillations was of the order of 5-15 Hz and the amplitude reached peak values of 1000 deg/sec or more. This component, due to either variations of maneuver induced rolling moments[11] or limit cycles caused by rolleron nonlinearities,[8] seemed to be too significant to be neglected.

The purpose of this paper is to extend the method of previous studies[9-12] successfully used for constant roll rates, to include periodic roll rate components. Analysis is carried out under the following assumptions: 1) the missile has two identical noninteracting guidance loops with linear time-invariant dynamics operating in perpendicular planes of maneuver; 2) the missile rolls about the axis defined by the intersection of these two planes at a rate which is a known periodic function of time; 3) the roll rate is sufficiently low so that the stability of the vehicle is not disturbed by inertial coupling or by Magnus effects; 4) line of sight angles, seeker gimbal angles, and angles of attack are negligibly small, i.e., body and wind axes both coincide with line of sight; 5) the missile is guided by proportional navigation and its trajectory is near to an ideal collision course; 6) missile and target velocities are constant; and 7) the missile time of flight is very large in comparison with equivalent first-order time constant of the guidance loop.

An analytical solution is obtained by: a) determining the effects due to periodic roll rate on the open loop characteristics of a two-channel control system; and b) investigating the behavior of the closed loop homing systems under the effects of the cross-coupling induced by periodic roll rate.

Rolling Two-Channel System

The dynamics of the perpendicularly oriented control channels of a cruciform homing missile are independently defined (see Assumption 1) in body coordinates and therefore are not affected by the roll. On the contrary, the equations of motion of a rolling airframe in body coordinates cannot be separated into the perpendicular planes of maneuver. Fortunately, it has been demonstrated[14] that in slender dynamically isotropic configurations, the effects of the roll are not observable in a nonrolling coordinate system.§ This fact, supported by Assumption 3, justifies one to consider the roll rate induced effects on the control system only.

Under Assumption 1, the state of each control channel is determined by a set of linear differential equations with constant coefficients.

$$\dot{x}_i = A_i x_i + b_i v_i \qquad w_i = c_i^T x_i \qquad i=1,2 \tag{1}$$

§For operational homing missiles, the ratio of axial and transversal moments of inertia is of the order of 0.01 or less.

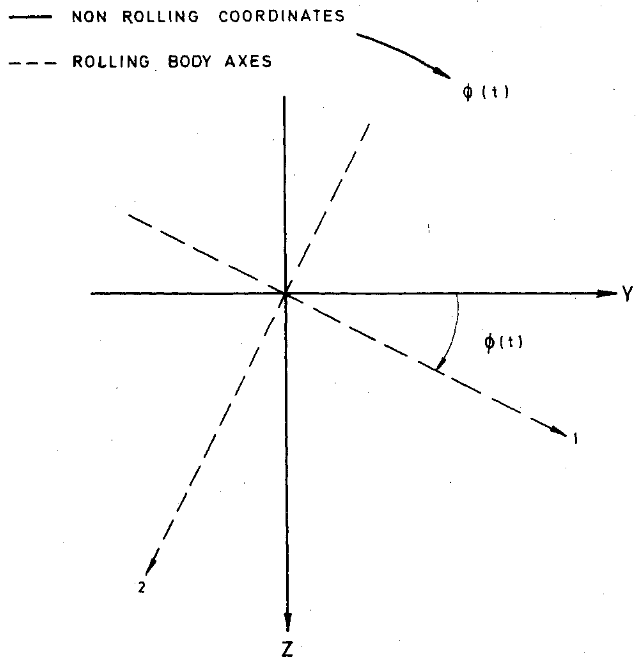

Fig. 1 Definition of the coordinate systems.

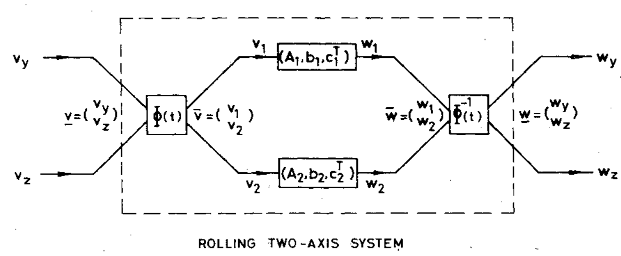

Fig. 2 Block diagram of a rolling two-channel system.

(The components of x_i consist, depending on the specific control system design, of quantities such as seeker head angular error, torquing moments, actuator signals, etc.) Two sets of equations (for $i=1,2$) can be combined to form a single one of higher dimension.

$$\dot{\bar{x}} = A\bar{x} + B\bar{v} \qquad \bar{w} = C\bar{x} \tag{2}$$

This two-channel system rotates with a periodically varying roll rate, which can be approximated, for the sake of simplicity, by

$$\dot{\phi}(t) = \dot{\alpha}_\phi + \dot{\beta}_\phi \cos(\omega_\phi t + \psi_\phi) \tag{3}$$

The coordinate systems of the rolling body and the nonrolling reference frame are defined in Fig. 1. The angular displacement between the two coordinate frames $\phi(t)$ determines the rotation matrix $\Phi[\phi(t)]$ which can also be expressed in the form

$$\Phi[\phi(t)] = \cos\phi(t) I - \sin\phi(t) J \tag{4}$$

where I is the two-dimensional identity matrix and

$$J \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

(Note that $J^2 = -I$.)

The components of the input and output vectors in the rolling frame are given by

$$\bar{v} = \Phi \underline{v} \qquad \bar{w} = \Phi \underline{w} \tag{6}$$

as shown in the block diagram of Fig. 2.

Substituting Eq. (6) into Eq. (2) results in

$$\dot{x} = Ax + B\Phi \underline{v} \qquad \Phi \underline{w} = Cx \tag{7}$$

The solution of Eq. (2) in body coordinates for zero initial conditions is

$$\bar{w}(t) = C\int_0^t \exp\{(t-\xi)A\}B\bar{v}(\xi)d\xi \tag{8}$$

Since both channels are identical, this can be written as

$$\bar{w}(t) = \int_0^t h(t-\xi)\bar{v}(\xi)d\xi \tag{9}$$

This solution can be transformed into the nonrolling coordinate system by

$$\underline{w}(t) = \Phi^{-1}[\phi(t)]\int_0^t h(t-\xi)\Phi[\phi(\xi)]\underline{v}(\xi)d\xi \tag{10}$$

Since $h(t-\xi)$ is a scalar function and $\Phi^{-1}[\phi(t)]$ is independent of ξ, Eq. (10) can be written as

$$\underline{w}(t) = \int_0^t h(t-\xi)\Phi^{-1}[\phi(t)]\Phi[\phi(\xi)]\underline{v}(\xi)d\xi \tag{11}$$

Using the relation $\Phi^{-1}(\delta)\Phi(\eta) = \Phi(\eta-\delta) = \Phi^{-1}(\delta-\eta)$, based on the well-known property of the rotation matrix, Eq. (11) can be rewritten as

$$\underline{w}(t) = \int_0^t h(t-\xi)\Phi^{-1}[\phi(t)-\phi(\xi)]\underline{v}(\xi)d\xi \tag{12}$$

For constant roll rate, this expression reduces to a simple convolution integral since

$$\phi(t) - \phi(\xi) = (t-\xi)\dot{\phi} \tag{13}$$

and in this case, a straightforward solution can be obtained by means of Laplace Transforms.

For the general case ($\dot{\phi} \neq$ const), it has been shown[11-12] that Eq. (12) can be expressed by an uniformly convergent series

$$\underline{w}(t) = \sum_{n=0}^{\infty} \Phi_n(t)\underline{g}_n(t) \tag{14}$$

where

$$\underline{g}_n(t) \triangleq (-1)^n \int_0^t \frac{(t-\xi)^n}{n!} h(t-\xi)\underline{v}(\xi)d\xi \tag{15}$$

$$\Phi_n(t) \triangleq a_n(t)I - b_n(t)J \tag{16}$$

$a_n(t)$ and $b_n(t)$ are related to the nth order derivatives of $\cos(t)$ and $\sin(t)$ by

$$\frac{d^n}{dt^n}[\cos\phi(t)] \triangleq a_n(t)\cos\phi(t) - b_n(t)\sin\phi(t)$$

$$\frac{d^n}{dt^n}[\sin\phi(t)] \triangleq b_n(t)\cos\phi(t) + a_n(t)\sin\phi(t) \tag{17}$$

The vector functions $\underline{g}_n(t)$ are convolution integrals having the Laplace Transforms

$$\underline{g}_n(s) = \frac{1}{n!}\frac{d^n h(s)}{ds^n}\underline{v}(s) \tag{18}$$

The formulation of Eq. (14) is of great help in the anlaytic discussion in the following sections.

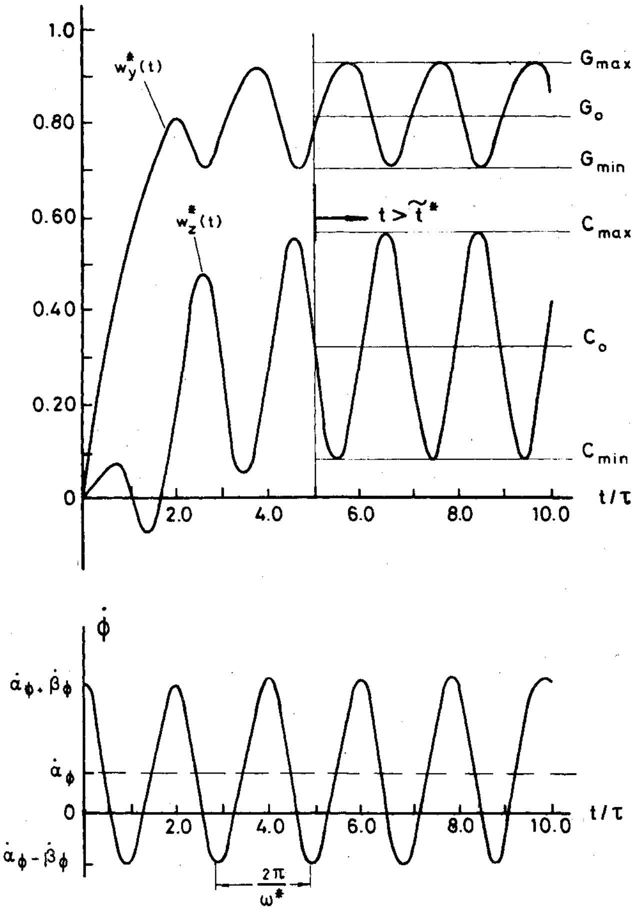

Fig. 3 Step response of two-channel system with periodic roll rate.

Steady-State Step Response for Periodic Roll Rate

For single axis asymptotically stable control systems, the steady-state (d.c.) gain is determined by the limit of the output for a unit step input, as $t\to\infty$. For a two-channel system, as described in the previous sections, two different constants have to be defined:

the "direct gain"

$$G \triangleq \lim_{t\to\infty} \frac{w_y}{\_\ulcorner v_y} = \lim_{t\to\infty} \frac{w_z}{\_\ulcorner v_z} \tag{19}$$

and the "cross-coupling factor"

$$C \triangleq \lim_{t\to\infty} \frac{w_z}{\_\ulcorner v_y} = -\lim_{t\to\infty} \frac{w_y}{\_\ulcorner v_z} \tag{20}$$

where $\_\ulcorner$ indicates unit step input at $t=0$.

Definitions of Eqs. (19) and (20) may be meaningless if the roll rate does not reach a limit for $t\to\infty$. In these cases (periodic roll rate is the most pertinent example) a different approach is required to obtain the steady-state step response. Defining $\underline{w}^*(t)$ as the output vector due to a unit step input $\_\ulcorner v_y$, its components are given by Eq. (12) as

$$w^*_y(t) = \int_0^t h(t-\xi)\cos[\phi(t)-\phi(\xi)]d\xi$$

$$w^*_z(t) = \int_0^t h(t-\xi)\sin[\phi(t)-\phi(\xi)]d\phi \tag{21}$$

These components are shown for an example of first-order dynamics and periodic roll rate in Fig. 3.

It can be clearly seen that after five to six time constants, the response of both channels is purely periodic (with the same period of the roll rate). The upper and lower bounds of the steady-state response can be also determined. This existence of periodic steady-state step response for periodic roll rate can be also determined analytically using Eqs. (14-18).

In an asymptotically stable system, the vector functions $g_n(t)$ in Eq. (15) approach, as $t\to\infty$, a limit for a step input, which can be directly calculated by the final value theorem of the Laplace Transforms. For Γv_y the z components are zero and the y components yield

$$\lim_{t\to\infty}[g_n(t)]_y = \lim_{s\to\infty}[sg_n(s)]_y = \frac{1}{n!}\frac{d^n h(s)}{ds^n}\Big|_{s=0} \triangleq h_n = \text{const.} \tag{22}$$

Thus, there exists for each n, a sufficiently large value $\bar{t}_n$ dependent on $\epsilon_n>0$ such that for all $t>\bar{t}_n$

$$|[g_n(t)]_y - h_n| < \epsilon_n \tag{23}$$

The components of $\underline{w}^*(t)$ can be expressed by Eqs. (14) and (10) as

$$w_y^*(t)=\sum_{n=0}^{\infty} a_n(t)[g_n(t)]_y \qquad w_z^*(t)=\sum_{n=0}^{\infty} b_n(t)[g_n(t)]_y \tag{24}$$

Define

$$G(t)\triangleq\sum_{n=0}^{\infty}a_n(t)h_n \qquad C(t)\triangleq\sum_{n=0}^{\infty}b_n(t)h_n \tag{25}$$

From the uniform convergence of Eq. (14) and the boundedness of $a_n(t)$ and $b_n(t)$ it follows, by Eq. (23-25), that there exists also a $\tilde{t}^*$ dependent on $\epsilon^*>0$ such that for all $t>\tilde{t}^*$.

$$|w_y^*(t)-G(t)|<\epsilon^* \tag{26}$$

and similarly

$$|w_z^*(t)-C(t)|<\epsilon^* \tag{27}$$

Thus for $t>\tilde{t}^*$, $G(t)$ and $C(t)$ represent the step response of the rolling two-channel system. These functions may therefore be called, by analogy to Eqs. (19) and (20), the "time-varying direct gain" and the "time-varying cross-coupling" functions, Since h_n $(n=0,1,2,...)$ are constants, the time dependence of $G(t)$ and $C(t)$ is due only to the variations of the roll rate through $a_n(t)$ and $b_n(t)$ respectively. Thus, periodic roll rate leads to periodic step response. The complete step response of a rolling system, and therefore the functions $G(t)$ and $C(t)$, can be computed for any given dynamics and time-varying roll rate by Eq. (21). However, direct computation provides only limited insight regarding the influence of the various parameters of the roll rate. To obtain this insight, an approximate analytical solution for $G(t)$ and $C(t)$ is developed in the next section.

The Approximate Solution

The input-output relationship of a two-channel rolling system, having only a first-order time-constant τ_1, is given by the vector equation[11]

$$\tau_1\dot{w}(t)+[I-\tau_1\dot{\phi}(t)J]w(t)=v(t) \tag{28}$$

The time-dependent functions $G(t)$ and $C(t)$ are the components of a particular solution for this equation with a unit step input in Γv_y. These functions have to be determined by solving for $t>0$ the following set of differential equations

$$\tau_1\dot{G}(t)+G(t)+\tau_1\dot{\phi}(t)C(t)=1$$
$$-\tau_1\dot{\phi}(t)G(t)+\tau_1\dot{C}(t)+C(t)=0 \tag{29}$$

For a periodic roll rate of a frequency ω_ϕ as given in Eq. (3), the solution will also be periodic with the same fundamental frequency. It can be expressed by its Fourier series

$$G(t)=G_0+\sum_{n=1}^{\infty}[G_n\cos(n\omega_\phi t)+K_n\sin(n\omega_\phi t)] = G_0+\sum_{n=1}^{\infty}\bar{G}_n\cos(n\omega_\phi t+\psi_n) \tag{30}$$

$$C(t)=C_0+\sum_{n=1}^{\infty}[C_n\cos(n\omega_\phi t)+D_n\sin(n\omega_\phi t)] = C_0+\sum_{n=1}^{\infty}\bar{C}_0+\sum_{n=1}^{\infty}\bar{C}_n\cos(n\omega_\phi t+\chi_n) \tag{31}$$

To compute the coefficients of the Fourier expansion, the following procedure, detailed in Appendix A, is carried out.

a) Substitute Eqs. (30), (31) and (3) into Eq. (29). b) Eliminate nonlinear terms using the identities

$$\cos(\omega t)\cos(n\omega t)=\tfrac{1}{2}\{\cos(n+1)\omega t+\cos(n-1)\omega t\}$$
$$\cos(\omega t)\sin(n\omega t)=\tfrac{1}{2}\{\sin(n+1)\omega t+\sin(n-1)\omega t\} \tag{32}$$

and c) Equate the coefficients of the respective harmonics.

As the set of Eqs. (A. 3-A.5) resulting from this procedure have four more unknowns than equations, an approximate solution has been tried by truncation of Eqs. (30) and (31). Here, the number of unknowns and equations are made equal and computations of the coefficients are possible. In Appendix A, an approximate solution of this type is presented, using the following nondimensional parameters

$$\alpha^*=\tau_1\dot{\alpha}_\phi \qquad \beta^*=\tau_1\beta_\phi \qquad \omega^*=\tau_1\omega_\phi \tag{33}$$

As shown in Eq. (A. 8-A. 14), this approximation, even in the simple case of $N=2$, yields expressions too complicated for convenient qualitative analysis.

By a further approximation, based on test data[13]

$$\omega^*>>\alpha^* \qquad \omega^*>>\beta^* \tag{34}$$

relatively simple formulas can be obtained for the coefficients of the Fourier expansions in Eqs. (30) and (31).

In the next section, it is shown that the important quantities needed to solve the homing equation of a rolling missile are

$$G_{\min}=\inf_t G(t) \qquad C_{\max}=\sup_t C(t) \tag{35}$$

Based on the assumption of Eq. (34), these limits yield

$$G_{\min}\simeq\frac{1-\alpha^*(\frac{\beta^*}{\omega^*})-\frac{1}{4}(\frac{\beta^*}{\omega^*})^2}{(1+\alpha^{*2})(1+\frac{\beta^{*2}}{2\omega^{*2}})} \tag{36}$$

$$C_{\max}\simeq\frac{\alpha^*+(\frac{\beta^*}{\omega^*})+\frac{\alpha^*}{4}(\frac{\beta^*}{\omega^*})^2}{(1+\alpha^{*2})(1+\frac{\beta^{*2}}{2\omega^{*2}})} \tag{37}$$

indicating that the effect of the periodic roll rate component depends on the parameter

$$\frac{\beta^*}{\omega^*}=\frac{\beta_\phi}{\omega_\phi}\triangleq\phi_p \tag{38}$$

which is the amplitude of the roll angle oscillations [as can be observed by integrating Eq. (3)].

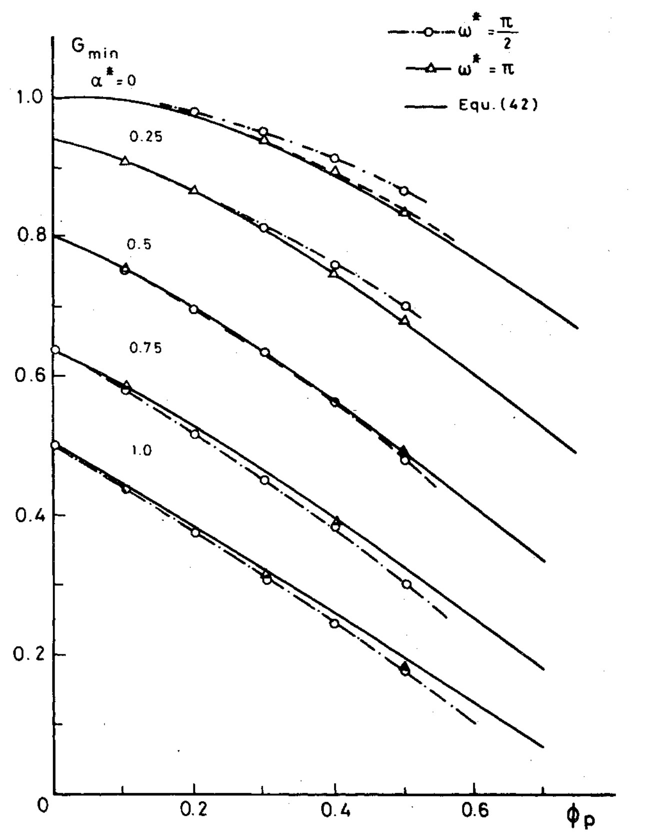

Fig. 4 Minimum value of the direct gain function for two-channel system with periodic roll rate.

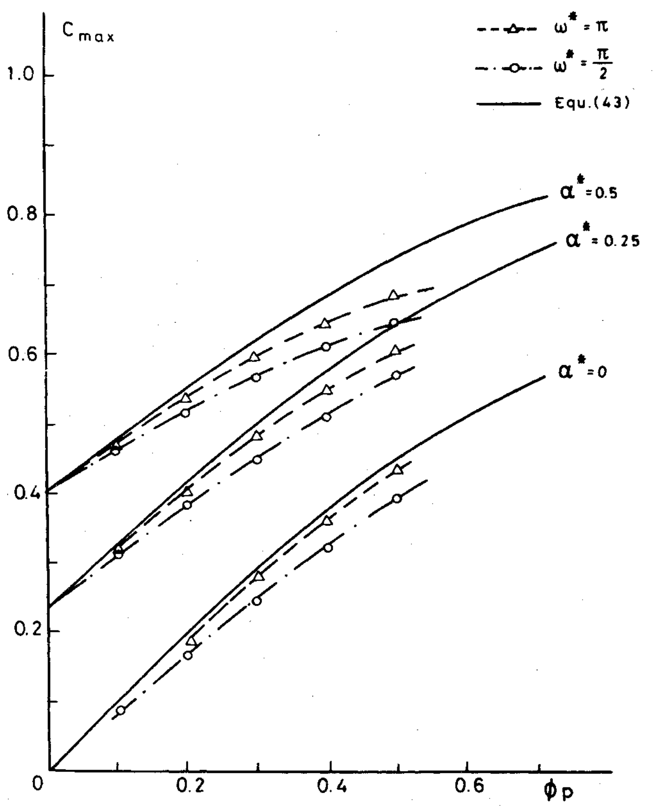

Fig. 5 Maximum value of cross-coupling function for two-channel system with periodic roll rate.

$$\phi(t)-\phi_0=\dot{\alpha}_\phi t+\frac{\dot{\beta}_\phi}{\omega_\phi}\sin(\omega_\phi t+\psi_\phi)$$

$$=\alpha^*(\frac{t}{\tau_I})+\phi_p\sin(\omega^*\frac{t}{\tau_I}+\psi_\phi) \qquad (39)$$

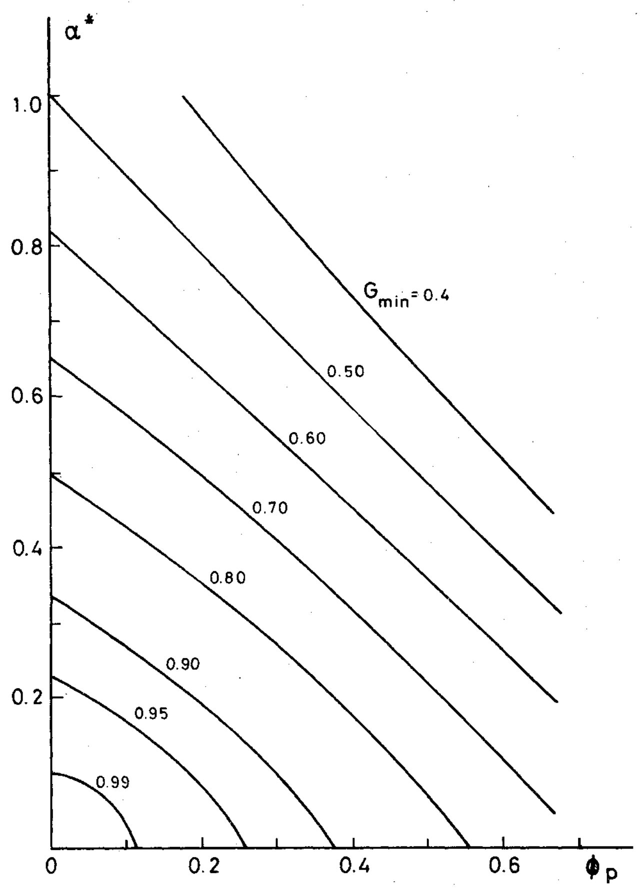

Fig. 6 Contours of constant G_{min} for two-channel system with periodic roll rate.

The agreement between the exact values of G_{min} obtained by Eq. (21) and the results of the approximate formulae of Eq. (36) is very good, as can be seen in Fig. 4. For amplitudes not exceeding $\phi_p=0.5$rd, the effect of the roll rate frequency (even for values as low as $\omega^*=\pi/2$) is not significant. The exact values of C_{max} derived by Eq. (21) are compared to the results of the approximation of Eq. (37) in Fig. 5. Although the agreement here is not as good as for G_{min}, this comparison confirms the dominant role of ϕ_p and the secondary influence of the frequency.

The simple relations in Eqs. (36) and (37) provide an insight into the influence of the various roll rate components and their interaction on the step response of a rolling two-channel system. It can be directly observed that due to the terms $(\alpha^*\beta^*/\omega^*)$, the effects cannot be separated. Moreover, Eq. (36) leads to determine, for any preassigned required value of G_{min}, the permissible value of the nondimensional average roll rate α^* for given amplitudes of roll oscillation ϕ_p. This is shown in Fig. 6 by the contours of constant values of G_{min} plotted in the $\alpha^*-\phi_p$ plane.

Homing of a Missile with Periodic Roll Rate

In Fig. 7, the scalar variables of two-dimensional proportional navigation are defined. In order to investigate the homing process of a rolling missile, the analysis was extended in previous works[9-12] to three dimensions. The vector equation of the line of sight rate for homing against a nonmaneuvering target, derived[9,12] under the assumptions detailed in the introduction, has the following form

$$\ddot{\underline{\lambda}}(t_0-t)+\{[aG(t)-2]I+aC(t)J\}\dot{\underline{\lambda}}=0 \qquad (40)$$

The initial condition, $\dot{\underline{\lambda}}_0$, depends on the launching error relative to the collision course. This equation yields the solution

$$\dot{\underline{\lambda}}(t)=\exp[\int_0^t D(t)dt]\dot{\underline{\lambda}}_0 \qquad (41)$$

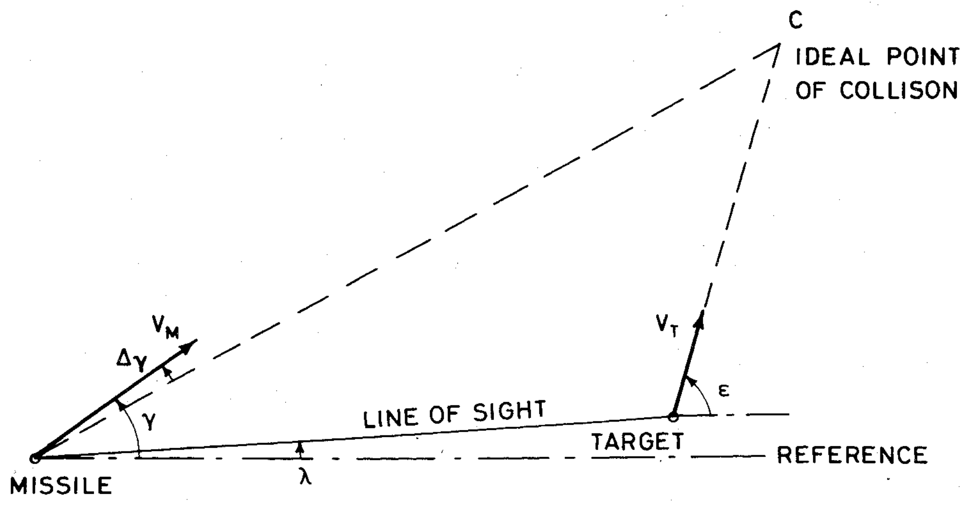

Fig. 7 Homing geometry in two-dimensions.

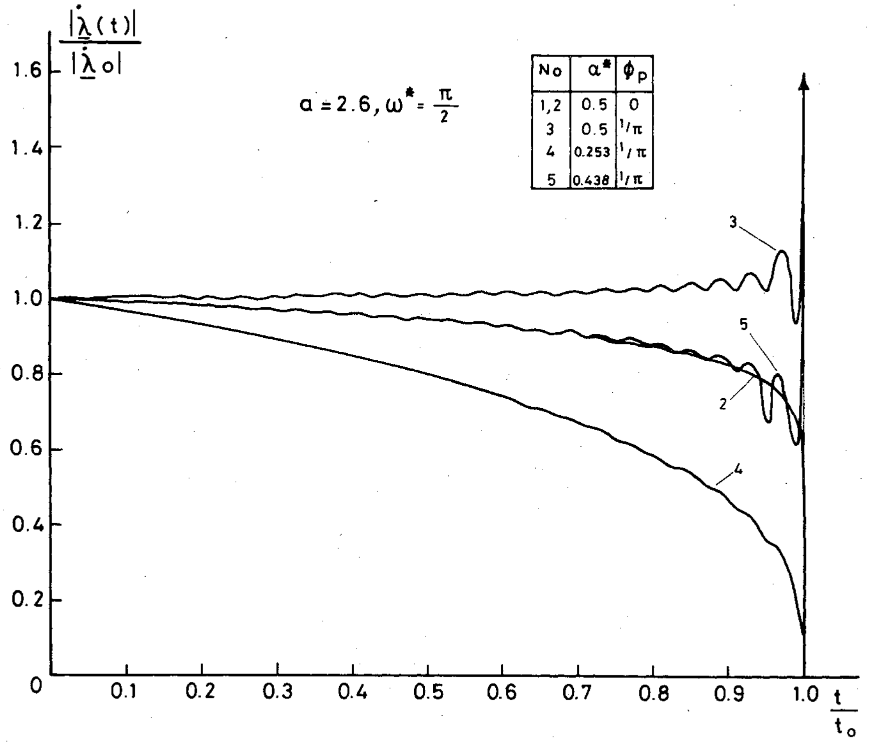

Fig. 8 Resultant line of sight rate of homing missiles with periodic roll rate.

Table 1 Summary of results

No.	a	α^*[rd]	β^*[rd]	ω^* [rd]	ϕ_p[rd]	G_0	$G_{\min}$	aG_0	$aG_{\min}$	Remarks
1	2.08	0	0	–	0	1	1	2.08	2.08	convergence
2	2.60	0.5	0	–	0	0.8	0.8	2.08	2.08	same behavior as No. 1
3	2.60	0.5	0.5	0.5π	$1/\pi$	0.76	0.62	1.98	1.61	divergence
4	2.60	0.253	0.5	0.5π	$1/\pi$	0.89	0.8	2.33	2.08	convergence
5	2.60	0.438	0.5	0.5π	$1/\pi$	0.8	0.67	2.08	1.73	divergence at intercept inspite of converging tendency

because the matrix

$$D(t) = \frac{1}{t_0 - t}\{[aG(T) - 2]I + aC(t)J\} \tag{42}$$

is commutative with its own integral

$$E(t) \triangleq \int_0^t D(\xi)\,d\epsilon = E_1(t)I + E_2(t)J \tag{43}$$

due to their special structure.

$$E_1(t) = \int_0^t \frac{aG(\xi) - 2}{t_0 - \xi}\,d\xi \tag{44}$$

$$E_2(t) = \int_0^t \frac{aC[\xi]}{t_0 - \xi}\,d\xi \tag{45}$$

Thus the solution of Eq. (41) has form of

$$\dot{\underline{\lambda}}(t) = \exp\{-E_1(t)\}\ [\cos E_2(t)I - \sin E_2(t)J]\dot{\underline{\lambda}}_0 \tag{46}$$

expressing a spiral behavior. The convergence of the solution depends only on E_1 and it requires that

$$\lim_{t \to t_0} [\exp\{-E_1(t)\}] = 0 \tag{47}$$

The spiral angular velocity of the vector $\underline{\dot{\lambda}}(t)$ in the plane perpendicular to the line of sight is determined by

$$E_2(t) = \frac{aC(t)}{t_0 - t} \leq \frac{aC_{\max}}{t_0 - t} \tag{48}$$

It is also easy to see that

$$\exp[-E_1(t)] \leq \left(\frac{t_0 - t}{t_0}\right)^{aG_{\min} - 2} \tag{49}$$

and therefore the condition of convergence will be satisfied if

$$aG_{\min} - 2 > 0 \tag{50}$$

Up to this point no use has been made of the periodic nature of the roll rate. For the type of a single harmonic roll rate given in Eq. (3), a closed form solution of Eq. (46) is obtained in Appendix B using the integral consine (Ci) and integral sine (Si) functions. The solution indicates that due to the asymptotic nature of these functions, the condition in Eq. (47) can be satisfied by

$$aG(t_0) - 2 > 0 \tag{51}$$

i.e., convergence is determined by the value of the "direct gain" function *at the moment of the intercept*. However, this improvement over Eq. (50) seldom can be of benefit because the phase of the periodic roll rate generally does not depend on the range, and therefore, the exact value of $G(t_0)$ cannot be predicted. Thus, for design purposes, the more restrictive limitation of Eq. (50), based on the minimum value (or lower limit) of $G(t)$, should be used. This limit can be computed with adequate precision by Eq. (36).

To determine the effect of the periodic component of the roll rate on the convergence of the line of sight rate in proportional homing against a nonmaneuvering target, a set of examples is presented. For the ratio of $t_0/\tau_1 = 100$ (obeying Assumption 7), five cases were computed and the results are summarized in Table 1 and Fig. 8. In these examples, the phase of the roll rate was chosen to that $G(t_0)$ would coincide approximately with $G_{\min}$, making conditions Eqs. (50) and (51) equivalent. It is interesting to observe that while the convergence at intercept is determined by the value of "$aG(t_0)$" bounded by "$aG_{\min}$", the behavior of the line of sight is dominated most of the time (excluding the terminal phase) by "aG_0", G_0 being the average value of $G(t)$ in Eq. (30).

Conclusions

In this paper, the analytic method of Refs. 9, 10, 12, leading to a closed form solution for homing missiles rolling at a constant rate, was extended to the case of periodic roll rate. The necessary condition for the convergence of the line of sight rate given in Eq. (50) requires the prediction of the minimum value of the "direct gain" function of the rolling guidance system. The approximate method developes in this paper provides a good estimate for this value and a clear insight into the mutual effect of the constant and periodic roll-rate components. The approximation also helps to demonstrate that the main effect of the periodic roll rate is due to the amplitude of the roll angle oscillation. The frequency, if sufficiently high, has a secondary influence. The analysis and examples show that the limitation on the maximum allowable average roll rate, imposed by the presence of a periodic component, is more severe than the limitation for a constant roll rate.

For the sake of simplicity, this paper presented only examples of first-order dynamics and single frequency roll rate. However, the analysis can be directly extended to dynamics of higher order and to any type of periodic time-varying roll rate.

Appendix A

Derivation of Approximate Formulae for the Coefficients of the Fourier Expansion of $G(t)$ and $C(t)$

The two different expressions for the Fourier expansions of $G(t)$ and $C(t)$ are given in Eqs. (30) and (31). Substituting the "cos and sin" formulae into Eq. (29) using the trigonometric identities in Eq. (32), leads to the expressions

$$\begin{aligned}
&\tau_1\omega_\phi \sum_{n=1}^{\infty} n[K_n\cos(n\omega_\phi t) - G_n\sin(n\omega_\phi t)] + G_0 \\
&+ \sum_{n=1}^{\infty} [G_n\cos(n\omega_\phi t) + K_n\sin(n\omega_\phi t)] \\
&+ \tau_1(\dot{\alpha}_\phi + \dot{\beta}_\phi\cos\omega_\phi t)C_0 + \tau_1\frac{\dot{\beta}_\phi}{2}C_1 \\
&+ \tau_1\dot{\alpha}_\phi(C_1\cos\omega_\phi t + D_1\sin\omega_\phi t) \\
&+ \tau_1\sum_{n=2}^{\infty}\{[\dot{\alpha}_\phi C_n + \frac{\dot{\beta}_\phi}{2}(C_{n-1} + C_{n+1})]\cos(n\omega_\phi t) \\
&+ [\dot{\alpha}_\phi D_n + \frac{\dot{\beta}_\phi}{2}(D_{n-1} + D_{n+1})]\sin(n\omega_\phi t)\} = 1
\end{aligned} \tag{A1}$$

$$\begin{aligned}
&\tau_1\omega_\phi \sum_{n=1}^{\infty} n[D_n\cos(n\omega_\phi t) - C_n\sin(n\omega_\phi t)] \\
&+ C_0 + \sum_{n=1}^{\infty}[C_n\cos(n\omega_\phi t) + D_n\sin(n\omega_\phi t)] \\
&- \tau_1(\dot{\alpha}_\phi + \dot{\beta}_\phi\cos\omega_\phi t)G_0 - \tau_1\frac{\dot{\beta}_\phi}{2}G_1 \\
&- \tau_1\dot{\alpha}_\phi(G_1\cos\omega_\phi t + K_1\sin\omega_\phi t) \\
&- \tau_1\sum_{n=2}^{\infty}\{[\dot{\alpha}_\phi G + \frac{\dot{\beta}_\phi}{2}(G_{n-1} + G_{n+1})]\cos(n\omega_\phi t) \\
&+ [\dot{\alpha}_\phi K_n + \frac{\dot{\beta}_\phi}{2}(K_{n-1} + K_{n+1})]\sin(n\omega_\phi t)\} = 0
\end{aligned} \tag{A2}$$

Equating the coefficient of the respective harmonics ($n = 0,1,2,\ldots$) and introducing the nondimensional variables defined in Eq. (33), yields the following set of algebraic equations

For $n = 0$

$$\begin{aligned}
&G_0 + \alpha^* C_0 + \frac{\beta^*}{2}C_1 = 1 \\
&\alpha^* G_0 + \frac{\beta^*}{2}G_1 - C_0 = 0
\end{aligned} \tag{A3}$$

For $n = 1$

$$\begin{aligned}
&\beta^* C_0 + G_1 + \alpha^* C_1 + \omega^* K_1 + \frac{\beta^*}{2}C_2 = 0 \\
&-\omega^* G_1 + K_1 + \alpha^* D_1 + \frac{\beta^*}{2}D_2 = 0 \\
&\beta^* G_0 + \alpha^* G_1 - C_1 - \omega^* D_1 + \frac{\beta^*}{2}G_2 = 0 \\
&\omega^* C_1 - D_1 + \alpha^* K_1 + \frac{\beta^*}{2}K_2 = 0
\end{aligned} \tag{A4}$$

For $n \geq 2$

$$G_n + n\omega^* K_n + \alpha^* C_n + \frac{\beta^*}{2}(C_{n-1} + C_{n+1}) = 0$$

$$K_n - n\omega^* G_n + \alpha^* D_n + \frac{\beta^*}{2}(D_{n-1} + D_{n+1}) = 0$$

$$-C_n - n\omega^* D_n + \alpha^* G_n + \frac{\beta^*}{2}(G_{n-1} + G_{n+1}) = 0$$

$$-D_n + n\omega^* C_n + \alpha^* K_n + \frac{\beta^*}{2}(K_{n-1} + K_{n+1}) = 0 \qquad \text{(A5)}$$

Due to the terms of Index 2 in Eq. (A4) and the terms of Index $(n+1)$ in Eq. (A5), the number of unknowns exceeds the number of equations by four. Therefore, this formulation does not have a solution based on recursive computation. However, by assuming that for $n \geq N$ the coefficients are zero

$$G_n = K_n = C_n = D_n = 0 \qquad \text{(A6)}$$

a set of $(4N-2)$ equations can be solved and an approximate solution is obtained.

For $N=1$, the solution disregards the periodic roll rate and the result obtained is identical to the case of constant roll rate.[10]

$$G = G_0 = \frac{1}{1+\alpha^{*2}} \qquad C = C_0 = \alpha^* G_0 = \frac{\alpha^*}{1+\alpha^{*2}} \qquad \text{(A7)}$$

For $N=2$ the set of six equations [Eqs. (A3) and (A4)] yields the following solution

$$G_0 = \frac{1}{1+\alpha^{*2}}\left[1 - \frac{\beta^{*2}}{2}\frac{(1+\alpha^{*2})(1+\omega^{*2} - 3\alpha^{*2} + [\beta^{*2}/2])}{\Delta_1}\right] \qquad \text{(A8)}$$

$$C_0 = \frac{\alpha^*}{1+\alpha^{*2}}\left[1 - \frac{\beta^{*2}}{2}\frac{(1+\alpha^{*2})(3+\omega^{*2} - \alpha^{*2} + [\beta^{*2}/2])}{\Delta_1}\right] \qquad \text{(A9)}$$

$$G_1 = \frac{-2(\alpha^*\beta^*)(1+\alpha^{*2})}{\Delta_1} \qquad \text{(A10)}$$

$$K_1 = -\frac{\omega^*\alpha^*\beta^*(3+\omega^{*2} - \alpha^{*2} + [\beta^{*2}/2])}{\Delta_1} \qquad \text{(A11)}$$

$$C_1 = \frac{\beta^*(1+\alpha^{*2})(1-\omega^{*2} - \alpha^{*2} + [\beta^{*2}/2])}{\Delta_1} \qquad \text{(A12)}$$

$$D_1 = \frac{-\omega^*\beta^*(1+\omega^{*2} - 3\alpha^{*2} + [\beta^{*2}/2])}{\Delta_1} \qquad \text{(A13)}$$

where Δ_1 is the determinant having the form

$$\Delta_1 = (1+\alpha^{*2})\left[(1+\omega^{*2})^2 - \omega^{*2}(2\alpha^{*2} - \beta^{*2}) + \alpha^{*2}(2+\alpha^{*2} - \beta^{*2}) + \frac{\beta^{*4}}{4}\right] \qquad \text{(A14)}$$

It is quite difficult to perform a qualitative analysis based on these complicated formulae. Moreover, the predicted aperiodic behavior of $G(t)$ for $\alpha^* = 0$ does not agree with the results of the direct computation of Eq. (23).

In order to gain insight into the effects of the roll rate parameters and to make this solution appropriate for qualitative analysis, it will be assumed that the frequency of the roll rate is high.

$$\omega^* >> \alpha^* \qquad \omega^* >> \beta^* \qquad \text{(A15)}$$

This assumption is based on flight test data evaluation.[13]

Under this assumption, the set of Eq. (A5) yields for $N=3$

$$K_2 \cong -\frac{\beta^*}{4\omega^*}C_1$$

$$G_2 \cong \frac{\beta^*}{4\omega^*}D_1$$

$$D_2 \cong \frac{\beta^*}{4\omega^*}G_1$$

$$C_2 \cong -\frac{\beta^*}{4\omega^*}K_1 \qquad \text{(A16)}$$

Substituting these terms into Eq. (A4), it can be seen that they are negligible relative to the terms multiplied by ω^*. They have, therefore, no effect on the coefficients of lower order. Carrying assumption of Eq. (15) further, it can be concluded from the second and fourth equations of Eq. (A4), that D_1 and K_1 have to be an order of magnitude larger than C_1 or D_1 (as both of them appear multiplied by ω^*). Thus the two other equations yield

$$K_1 \cong -\frac{\beta^*}{\omega^*}C_0 \qquad D_1 \cong \frac{\beta^*}{\omega^*}G_0 \qquad \text{(A17)}$$

and the previous pair leads to

$$C_1 = \frac{D_1 - \alpha^* K_1}{\omega^*} \cong \frac{\beta^*}{\omega^{*2}}[G_0 + \alpha^* C_0]$$

$$G_1 = \frac{K_1 + \alpha^* D_1}{\omega^*} \cong \frac{\beta^*}{\omega^{*2}}[\alpha^* G_0 - C_0] \qquad \text{(A18)}$$

From these expressions, including Eq. (A16), it can be observed that the main effect of the periodic roll rate is due to the factor β^*/ω^*, which is the amplitude of the roll angle oscillations. Using the definition

$$\phi_p \triangleq \frac{\beta^*}{\omega^*} = \frac{\dot{\beta}_\phi}{\omega_\phi} \qquad \text{(A19)}$$

and substituting Eq. (A18) into Eq. (A3), the following results are obtained:

$$G_0 = [(1+\alpha^{*2})(1+\frac{\phi_p^2}{2})]^{-1}; \quad C_0 = \alpha^* G_0 \qquad \text{(A20)}$$

The formulas for the coefficients of the first order are

$$G_1 = 0, \qquad C_1 = \phi_p\left(\frac{1+\alpha^{*2}}{\omega^*}\right)G_0$$

$$K_1 = -\phi_p \alpha^* G_0, \qquad D_1 = \phi_p G_0 \qquad \text{(A21)}$$

and the coefficients of the second order have the form

$$G_2 = \frac{\phi_p^2}{4}G_0, \qquad C_2 = \frac{\phi_p^2}{4}\alpha^* G_0$$

$$K_2 = -\frac{\phi_p^2}{4}\frac{(1+\alpha^{*2})}{\omega^*}G_0, \qquad D_2 = 0 \qquad \text{(A22)}$$

The amplitude of the first harmonic of $G(t)$ is

$$\bar{G}_1 = |K_1| = \phi_p \alpha^* G_0 \qquad \text{(A23)}$$

and its phase (due to the negative sign of K_1) relative to the roll rate,

$$\psi_1 = \frac{\pi}{2} \tag{A24}$$

The second harmonic of $G(t)$ has an amplitude of

$$\bar{G}_2 = \frac{\phi_p^2}{4} G_0 \left[1 + \left(\frac{1+\alpha^{*2}}{\omega^*}\right)\right]^{1/2} \cong \frac{\phi_p^2}{4} G_0 \tag{A25}$$

and a phase of

$$\psi_2 = tg^{-1}\left(\frac{K_2}{G^2}\right) = tg^{-1} \frac{(1+\alpha^{*2})}{\omega^*} << 1 \tag{A26}$$

Thus, the approximate formula for $G(t)$, under assumption of Eq. (A15), is

$$G(t) \approx G_0 \left[1 - \alpha^* \phi_p \sin\omega_\phi t + \frac{\phi_p^2}{4} \cos(2\omega_\phi t)\right] \tag{A27}$$

Similarly, the coefficients of the cross-coupling function are

$$\bar{C}_1 = G_0 \phi_p \left[1 + \left(\frac{1+\alpha^{*2}}{\omega^*}\right)^2\right]^{1/2} \approx G_0 \phi_p \tag{A28}$$

$$\bar{\chi}_1 = tg^{-1}\left(\frac{-D_1}{C_1}\right) = -\frac{\pi}{2} + \psi_2 \approx -\frac{\pi}{2} \tag{A29}$$

$$\bar{C}_2 = G_0 \alpha^* \frac{\phi_p^2}{4}; \qquad \chi_2 = 0 \tag{A30}$$

and

$$C(t) \approx G_0 \left[\alpha^* + \phi_p \sin\omega_\phi t + \alpha^* \frac{\phi_p^2}{4} \cos(2\omega_\phi t)\right] \tag{A31}$$

Based on Eq. (A27), the minimum value of $G(t)$ can be approximated by

$$G_{\min} \approx G_0 - \bar{G}_1 - \bar{G}_2 = G_0 \left(1 - \alpha^* \phi_p - \frac{\phi_p^2}{4}\right) \tag{A32}$$

using the value of G_0 given by Eq. (A20).

The maximum value of the cross-coupling function is determined in the same way by

$$C_{\max} \approx C_0 + \bar{C}_1 + \bar{C}_2 = G_0 \left(\alpha^* + \phi_p + \alpha^* \frac{\phi_p^2}{4}\right) \tag{A33}$$

Appendix B

Closed Form Solution of Eq. (46) for Periodic Roll Rate

To obtain a closed form solution of the line of sight rate $\dot{\lambda}$ for a proportional homing missile with periodic roll rate given by Eq. (3), the functions $E_1(t)$ and $E_2(t)$ in Eq. (44) and (45) have to be calculated.

$$E_1(t) = a \int_0^t \frac{G(\xi)}{t_0 - \xi} d\xi - 2 \int_0^t \frac{d\xi}{t_0 - \xi} \tag{A34}$$

$$E_2(t) = a \int_0^t \frac{C(\xi)}{t_0 - \xi} d\xi \tag{A35}$$

It has been shown in Appendix A that the time-varying functions, $G(t)$ and $C(t)$, can be approximated for periodic roll rate with adequate precision by

$$G(t) = G_0 + \bar{G}_1 \cos(\omega_\phi t + \psi_1) + \bar{G}_2 \cos(2\omega_\phi t + \psi_2) \tag{A36}$$

$$C(t) = C_0 + \bar{C}_1 \cos(\omega_\phi t + \chi_1) + \bar{C}_2 \cos(2\omega_\phi t + \chi_2) \tag{A37}$$

Integrals of the form

$$I_n = \int_0^t \frac{\cos(n\omega_\phi \xi + \psi_n)}{t_0 - \xi} d\xi \tag{A38}$$

can be analytically evaluated using the integral cosine $Ci(x)$ and the integral sine $Si(x)$ functions.[15]

These functions have an asymptotic behavior for large values of x

$$\lim_{x\to\infty} Si(x) = \pi/2 \qquad \lim_{x\to\infty} Ci(x) = 0 \tag{A39}$$

and for small values of the arguement $(x \to 0)$ they can be approximated by

$$\lim_{x\to 0} Si(x) = x \qquad \lim_{x\to 0} Ci(x) = \ln Kx \tag{A40}$$

where $\ln K = 0.577$.

Introducing new variables in Eq. (A33)

$$x_n \triangleq n\omega_\phi (t_0 - \xi); \quad x_0 \triangleq \omega_\phi t_0; \quad \delta_n \triangleq n\omega_\phi t_0 + \psi_n \tag{A41}$$

yields the expression

$$\begin{aligned} I_n &= \cos\delta_n \int_{n(x_0-\omega_\phi t)}^{nx_0} \frac{\cos x}{x} dx + \sin\delta_n \int_{n(x_0-\omega_\phi t)}^{nx_0} \frac{\sin x}{x} dx \\ &= \cos\delta_n [Ci(nx_0) - Ci(nx_0 - n\omega_\phi t)] \\ &\quad + \sin\delta_n \; [Si(nx_0) - Si(nx_0 - n\omega_\phi t)] \end{aligned} \tag{A42}$$

For large values of t_0 (see Assumption 7)

$$\begin{aligned} \lim_{t_0\to\infty} I_n &= -\cos\delta_n (Ci(nx_0 - n\omega_\phi t)] \\ &\quad + \sin\delta_n [\frac{\pi}{2} - Si(nx_0 - n\omega_\phi t)] \triangleq I_{n_0} \end{aligned} \tag{A43}$$

As the missile approaches the target (i.e., $t \to t_0$), the asymptotic values in Eq. (A35) can be used, and in this case, for large values of t_0

$$\lim_{t\to t_0} I_{n_0} = -\cos\delta_n [\ln Knx_0 (1 - \frac{t}{t_0})] + \sin\delta_n (\frac{\pi}{2}) \tag{A44}$$

Substituting Eq. (A42) into Eqs. (44) and (45), the complete closed form solution is obtained. For sufficiently long flight times, Eq. (A42) can be replaced by Eq. (A43).

From the closed form solution, the condition of convergence given in Eq. (47) can be determined. Substituting Eq. (A44) into Eq. (A34), it can be written that

$$\begin{aligned} \lim_{t\to t_0} E_1(t) &= -(aG_0 - 2)\ln(1 - \frac{t}{t_0}) \\ &\quad - a\bar{G}_1 \{\cos\delta_1 [\ln(1 - \frac{t}{t_0}) + \ln Kx_0] - \frac{\pi}{2}\sin\delta_1\} \\ &\quad - a\bar{G}_2 \{\cos\delta_2 [\ln(1 - \frac{t}{t_0}) + \ln 2Kx_0] - \frac{\pi}{2}\sin\delta_2\} \end{aligned} \tag{A45}$$

Regrouping the logarithmic time-dependent terms yields

$$\begin{aligned} \lim_{t\to t_0} E_1(t) &= \ln(1 - \frac{t}{t_0})[-a(G_0 + \bar{G}_1\cos\delta_1 \\ &\quad + \bar{G}_2\cos\delta_2) + 2] - a\{\bar{G}_1 [\ln(Kx_0)\cos\delta_1 - \frac{\pi}{2}\sin\delta_1] \\ &\quad + \bar{G}_2 [\ln(2Kx_0)\cos\delta_2 - \frac{\pi}{2}\sin\delta_2]\} \end{aligned} \tag{A46}$$

Recalling the definitions of $G(t)$ in Eq. (A36) and of δ_n in Eq. (A41), the expression for $E_1(t_0)$ becomes

$$\lim_{t \to t_0} E_1(t) = \ln(1 - \frac{t}{t_0})[-aG(t_0) + 2] - aK_0 \tag{A47}$$

where K_0 is the following constant

$$K_0 = \bar{G}_1 [\ln(Kx_0)\cos\delta_1 - \frac{\pi}{2}\sin\delta_1]$$

$$+ \bar{G}_2 [\ln(2Kx_0)\cos\delta_2 - \frac{\pi}{2}\sin\delta_2] \tag{A48}$$

Thus Eq. (47) can be written as

$$\lim_{t \to t_0} [\exp\{-E_1(t)\}] = e^{aK_0}\left(1 - \frac{t}{t_0}\right)^{aG(t_0) - 2} = 0 \tag{A49}$$

as aK_0 is a constant, Eq. (A49) implies that

$$aG(t_0) - 2 > 0 \tag{A50}$$

This indicates that convergence of the line of sight rate depends on the value of the direct gain function at the time of the intercept. This implies, using results of Appendix A for periodic roll rate and first-order dynamics, that

$$aG_0\left[1 - \alpha^*\phi_p \sin\omega_\phi t_0 + \frac{\phi_p^2}{4}\cos(2\omega_\phi t_0 + \frac{1 + \alpha^{*2}}{\omega^*})\right] > 2 \tag{A51}$$

where G_0, the average direct gain, is given by Eq. (20).

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